For now, we will focus on systems with n eqns & n variables

Gaussian Elimination

An efficient way to systematically solve a system of n equations with n variables is using Elimination to put system in upper triangular form (U.T.) and then back substitution.

Ex $\begin{aligned} x-y &= 1 \\ 2x+3y &= 3 \end{aligned}$ → Replace Eqn 2 by Eqn 2 − 2(Eqn 1)

↙ UT system

$\begin{aligned} x-y &= 1 \\ 5y &= 1 \end{aligned}$ → Solve 2nd eqn for y & plug into 1st equation to solve for x.

$\begin{aligned} x &= 6/5 \\ y &= 1/5 \end{aligned}$

We can encode this in an augmented matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & 3 \end{array}\right] \xrightarrow{\text{Replace R2 with } R2-2\cdot R1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 5 & 1 \end{array}\right] \rightarrow \begin{aligned} x-y &= 1 \\ 5y &= 1 \end{aligned} \rightarrow \begin{aligned} x &= 6/5 \\ y &= 1/5 \end{aligned}$$

↑ Coefficient Matrix

↑ UT matrix (square matrix with zeros below diagonal)

Pivots:

In $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & 1 \end{array}\right]$, the $(1,1)$-position and $(2,2)$-positions are called pivot positions. These are the first entries in their row that are not zero after putting in U.T. form.

The algorithm:

- Use R1 to create zeros below the first pivot
- Use the row with the 2nd pivot to create zeros below the 2nd pivot
- Continue in this way (if possible) until we reach an upper triangular matrix:

Ex: $2x + 4y - 2z = 2$

$4x + 9y - 3z = 8$

$-2x - 3y + 7z = 10$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{\text{Replace R2 with } R2 - 2R1} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

$$\downarrow \text{ Replace R3 with } R1 - (-R3)$$

Pivot positions

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right] \xleftarrow{\text{Replace R3 with } R3 - R1} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$\Rightarrow$

$2x + 4y - 2z = 2$

$y + z = 4$

$4z = 8$

Now back substitute:

$z = 2 \Rightarrow y = 2 \Rightarrow x = -1$

So soln is $(-1, 2, 2)$.

In the above exs, we see that there is a unique solution if we have a pivot in every row of the coefficient matrix. What if there are 0 or infinitely many solutions?

Ex: System with no sol'n

$$\begin{aligned} x-y&=1 \\ x-y&=2 \end{aligned} \longrightarrow \begin{aligned} x-y&=1 \\ 0&=1 \end{aligned} \Rightarrow \text{No solution}$$

Matrix View $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$ ← pivot in last column

Ex: System with infinite solutions

$$\begin{aligned} x-y&=1 \\ 2x-2y&=2 \end{aligned} \longrightarrow \begin{aligned} x-y&=1 \\ 0&=0 \end{aligned} \Rightarrow$$ All points satisfying $x-y=1$ are sol'ns. i.e. infinitely many

Matrix View $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ ← No pivot in last column + row with no pivot

In general,

- If there is a pivot in the final (augmented) column, then there is no solution.
- If the final column has no pivot:
  - If each row of the coefficient matrix has a pivot, then there is a unique solution.
  - If there is a row without a pivot, then there are infinitely many solutions.

Gaussian elimination can fail in putting a system in U.T. form. In this case, swapping rows is necessary to get back on track:

Ex: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{array}\right]$ $\xrightarrow[\text{Replace R3 with R3-2R1}]{\text{Replace R2 with R2-2R1}}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & -2 & -1 & 1 \end{array}\right]$ $\xrightarrow{\text{Swap R2 ↔ R3}}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & -3 & -1 \end{array}\right]$

Gauss-Jordan Elimination

Instead of doing backward substitution, we can continue performing operations on the augmented matrix until it is in so-called "row-reduced echelon form". In this form, we can easily read off the solution (if it is unique)

A matrix is in row-reduced echelon form (RREF) if

- All zero rows are at the bottom
- The leading nonzero entry of a row is to the right of the leading nonzero entry of the previous row.
- Each leading entry (pivot) is 1
- Each pivot is the only nonzero entry in its column

Ex: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are in RREF

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not in RREF

To put a matrix in RREF, we use:

Row Operations:

1.) Replacement: Add a multiple of one row to another

2.) Scaling: Multiply a row by a nonzero scalar.

3.) Swap: Swap two rows

The Algorithm

a) Use (1) (and possibly (3)) to make all entries below the pivots 0 (work left to right)
(ie. do Gaussian elimination)

b) Use (1) to make all entries above the pivots 0 (work from right to left)

b) use (2) to make pivot entries all 1

Ex:

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{\text{Last Example}} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Make 0

Replace R1 with $R2 + \frac{1}{2}R3$

Replace R2 with $R2 - \frac{1}{4}R3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & 8 \end{array}\right] \xleftarrow{\text{Replace R1 with R1 - 4R2}} \left[\begin{array}{ccc|c} 2 & 4 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Make 0

Multiply R1 by $\frac{1}{2}$

Multiply R3 by $\frac{1}{4}$

RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longrightarrow \begin{array}{l} x = -1 \\ y = 2 \\ z = 2 \end{array}$$

I

Solution!

Conclusion: If there is a unique solution, Gauss-Jordan turns $[A|b]$ into $[I|x]$, where x = solution

Key Point: the RREF of a matrix is unique.

So the order in which you perform row operations to get RREF doesn't matter. The algorithm above puts it in RREF in an efficient, systematic way.